"ENTROPIC" SOLUTIONS FOR TWO-PHASE FLUIDS FLOWS, PHASE TRANSITIONS, AND DAMAGE

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1 Abstract

In the recent publications [2] and [13], we have introduced a weak notion of solution, called "*entropic solution*", of diffuse interface models for applications relevant phenomena like the evolution of two viscous incompressible fluids and phase transitions and damage in thermoviscoelastic materials. This solvability concept reflects the basic principles of thermomechanics, as well as the thermodynamical consistency of the models. It allows us to obtain *global-in-time* existence theorems without imposing any restriction on the size of the initial data.
Key words: damage, phase transitions, thermoviscoelasticity, Navier-Stokes, Cahn-Hilliard equation, incompressible non-isothermal binary fluid, global-in-time existence of weak solutions.

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2 Introduction

In two recent contributions in cooperation with M. Eleuteri, Riccarda Rossi, and Giulio Schimperna (cf. [2, 13]) we reinterpret in the framework of diffuse interface models for two-phase fluid flows and phase transitions and damage in thermoviscoelastic materials, the concept of weak solution satisfying a suitable energy conservation and entropy inequality, recently introduced by E. Feireisl for a problem of heat conduction in fluids. This solution notion has also been exploited in the study of solid-liquid phase transition models with microscopic movements in [6] and, more recently, in [5, 7, 8], in the study of the evolution of liquid crystals flows. These ideas turn out to be particularly useful in the analysis of highly nonlinear and possibly degenerating PDEs, arising from different types of phase transitions and liquid flows models. Here, we would like to describe the application of the "*entropic solution*" notion to the case of phase transitions and damage in thermoviscoelastic materials and two-phase fluid flows.

Phase transitions and damage in thermoviscoelasticity. In [13] we analyze a PDE system modelling (non-isothermal) phase transitions and damage phenomena in thermoviscoelastic materials in a material body occupying a reference domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$. The system, derived in [12] according to M. Frémond's modeling approach (cf. [9, 10]), is:

$$\vartheta_t + \chi_t \vartheta + \rho\vartheta \operatorname{div}(\mathbf{u}_t) - \operatorname{div}(\mathsf{K}(\vartheta)\nabla\vartheta) = g + a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2 \quad \text{in } \Omega \times (0,T), \tag{1}$$

$$\mathbf{u}_{tt} - \operatorname{div}(a(\chi)\mathbb{V}\varepsilon(\mathbf{u}_t) + b(\chi)\mathbb{E}\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f} \quad \text{in } \Omega \times (0,T), \tag{2}$$

$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \operatorname{div}(|\nabla\chi|^{p-2}\nabla\chi) + W'(\chi) \ni -b'(\chi)\frac{\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u})}{2} + \vartheta \quad \text{in } \Omega \times (0,T), \tag{3}$$

supplemented with the boundary conditions (here n denotes the outward unit normal to $\partial\Omega$)

$$\mathsf{K}(\vartheta)\nabla\vartheta \cdot n = h, \qquad \mathbf{u} = 0, \qquad \partial_n\chi = 0 \qquad \text{on } \partial\Omega \times (0,T). \tag{4}$$

Here the unknowns ϑ and $\mathbf{u}$ respectively denote the absolute temperature of the system and the small displacement vector, while χ is an internal parameter: its meaning depends on the phenomenon described by (1)–(3) as well as the choices of the coefficients a and b in the momentum equation (2), and of the constant $\mu \in \{0,1\}$ in (3).

In particular, the choices $a(\chi) = 1 - \chi$ and $b(\chi) = \chi$ correspond to the case of *phase transitions* in thermoviscoelastic materials: in this case, χ is the order parameter, standing for the local proportion of one of the two phases taking values between 0 and 1, $\chi = 1$ stands for the liquid phase while $\chi = 0$ for the solid one and one has $0 < \chi < 1$ in the so-called *mushy regions*. Irreversibility of the phase transition process may be encompassed in the model by taking $\mu = 1$ in (3), which "activates" the term $\partial I_{(-\infty,0]}(\chi_t)$ (i.e. the subdifferential in the sense of convex analysis of the indicator function $I_{(-\infty,0]}$, evaluated at χ_t), yielding the constraint $\chi_t \leq 0$ a.e. in $\Omega \times (0,T)$. The meaning of $a(\chi) = 1 - \chi$ and $b(\chi) = \chi$ in (2) is that, in the *purely* solid phase $\chi = 0$ only the elastic energy, in addition to the thermal expansion energy, contributes to the stress $\sigma = a(\chi)\mathbb{V}\varepsilon(\mathbf{u}_t) + b(\chi)\mathbb{E}\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}$ (where $\mathbb{E}$ and $\mathbb{V}$ are the elasticity and viscosity tensors, respectively). Instead, in the *purely* liquid, or "viscous", phase $\chi = 1$ only the viscosity contribution remains, whereas in mushy regions both elastic and viscous effects are present.

The choices $a(\chi) = b(\chi) = \chi$ correspond to *damage*. In this case, χ is the damage parameter, assessing the soundness of the material microscopically, around a point in the material domain Ω. We assume $\chi = 0$ in case of complete damage, while χ takes the value 1 when the material is completely undamaged, and $0 < \chi < 1$ describes *partial damage*.

The function K in (1) is the heat conductivity, W in (3) is a mixing energy density, which we assume of the form $W = \widehat{\beta} + \widehat{\gamma}$ with $\widehat{\beta} : \mathrm{dom}(\widehat{\beta}) \to \mathbb{R}$ convex, possibly nonsmooth, and $\widehat{\gamma} \in \mathrm{C}^2(\mathbb{R})$, while $\mathbf{f}$ is a given bulk force, and g and h heat sources. The p-Laplacian term in (3) reflects the fact that we are within a *gradient theory* for phase transitions and damage.

The model is thermodynamically consistent: in particular, no *small perturbation assumption* is adopted, which results in the presence of quadratic terms on the right-hand side of the temperature equation, only estimated in L^1. Moreover, the whole system has a highly nonlinear character. Hence, we address in [13] the existence of a weak notion of solution, referred to as "*entropic*", where the temperature equation is formulated with the aid of an entropy inequality, and of a total energy inequality. This solvability concept, reflecting the basic principles of thermomechanics, allows us to obtain *global-in-time* existence theorems without any restriction on the size and any further regularity requirements of the initial data.

Two-phase fluids flows. In [2] we introduce a diffuse interface model describing the evolution of a mixture of two different viscous incompressible fluids of equal density. The main novelty of [2] consists in the fact that the effects of temperature on the flow are taken into account. In the mathematical model, the evolution of the velocity $\mathbf{v}$ is ruled by the Navier-Stokes system with temperature-dependent viscosity, while the order parameter φ representing the concentration of one of the components of the fluid is assumed to satisfy a convective Cahn-Hilliard equation. The effects of the temperature are prescribed by a suitable form of the heat equation. Namely, we consider the following equations:

$$\mathrm{div}\,\mathbf{v} = 0, \tag{5}$$

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla_x\mathbf{v} + \nabla_x p = \mathrm{div}\,\mathbb{S} - \varepsilon\mathrm{div}(\nabla_x\varphi\otimes\nabla_x\varphi), \qquad \mathbb{S} = \nu(\theta)D\mathbf{v}, \tag{6}$$

$$\varphi_t + \mathbf{v}\cdot\nabla_x\varphi = \Delta\mu, \tag{7}$$

$$\mu = -\varepsilon\Delta\varphi + \frac{1}{\varepsilon}F'(\varphi) - \theta, \tag{8}$$

$$c_V(\theta)\theta_t + c_V(\vartheta)\mathbf{v}\cdot\nabla_x\theta + \theta(\varphi_t + \mathbf{v}\nabla\varphi) - \mathrm{div}(\kappa(\theta)\nabla_x\theta) = \nu(\theta)|D\mathbf{v}|^2 + |\nabla_x\mu|^2, \tag{9}$$

in $\Omega\times(0,T)$, being Ω a bounded and sufficiently regular subset of $\mathbb{R}^3$ and $T > 0$ a given final time. Here p is the pressure, $\mathbb{S} = \nu(\theta)D\mathbf{v}$ represents the dissipative part of the stress tensor, where $D\mathbf{v} = (\nabla_x\mathbf{v} + \nabla_x^t\mathbf{v})/2$, and $\nu(\theta) > 0$ is the viscosity of the mixture. Moreover, $c_V(\theta)$ stands for the specific heat, $\kappa(\theta)$ indicates the heat conductivity, $\varepsilon > 0$ is a (small) parameter related to the "thickness" of the interfacial region, and μ is an auxiliary variable (usually named chemical potential) which helps particularly for the statement of the weak formulation of the model. Finally, $F(\varphi)$ is some suitable mixing energy density whose expression is specified below. The capillarity forces due to surface tension are modeled by an extra-contribution $\varepsilon\nabla_x\varphi \otimes \nabla_x\varphi$ in the global stress tensor appearing in the right-hand side of (5). The Cahn-Hilliard and temperature equations will be complemented by no-flux conditions, while the velocity $\mathbf{v}$ will be assumed to satisfy the so-called *complete slip* conditions.

However, because of the presence of the quadratic forcing terms in (9), this equation has to be replaced, in the weak formulation, by an equality representing energy conservation complemented with a differential inequality describing production of entropy. The main advantage of introducing this notion of solution is that, while the thermodynamical consistency is preserved, at the same time the "*entropic formulation*" is more tractable mathematically. Indeed, global-in-time existence for the initial-boundary value problem associated to the weak formulation of the model is proved by deriving suitable a-priori estimates and showing weak sequential stability of families of approximating solutions. Actually, it is not difficult to prove that, at least for *sufficiently smooth* weak solutions, the total energy balance together with the entropy inequality imply the original form of the heat equation (9). On the other hand, since this regularity in our case is not at all known (for instance due to the occurrence of the 3D Navier-Stokes system), this notion of solution turns out to be particularly useful because it permits to prove a global in time existence result in 3D, which was not known in the literature.

Better regularity properties are expected to hold for weak solutions in the 2D case. This is the subject of a the more recent paper [3], where we also analyze the long-time dynamics of the model.

3 Main results

3.1 The main results in on phase transitions and damage

The "entropic formulation". In order to get a global-in-time existence result for the PDE system (1)–(3) we need to exploit a weak solution notion where, as we have already explained in the Introduction, the temperature equation (1), featuring quadratic terms on its right-hand side, has to be weakly formulated in terms of an *entropy inequality*

$$\begin{aligned}
&\int_s^t\int_\Omega (\log(\vartheta)+\chi)\varphi_t \,\mathrm{d}x\,\mathrm{d}r + \rho\int_s^t\int_\Omega \mathrm{div}(\mathbf{u}_t)\varphi\,\mathrm{d}x\,\mathrm{d}r - \int_s^t\int_\Omega \mathsf{K}(\vartheta)\nabla\log(\vartheta)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}r\\
&\leq \int_\Omega (\log(\vartheta(t))+\chi(t))\varphi(t)\,\mathrm{d}x - \int_\Omega (\log(\vartheta(s))+\chi(s))\varphi(s)\,\mathrm{d}x - \int_s^t\int_\Omega \mathsf{K}(\vartheta)\frac{\varphi}{\vartheta}\nabla\log(\vartheta)\cdot\nabla\vartheta\,\mathrm{d}x\,\mathrm{d}r \qquad (10)\\
&\quad - \int_s^t\int_\Omega \left(g + a(\chi)\varepsilon(\mathbf{u}_t)\mathbb{V}\varepsilon(\mathbf{u}_t) + |\chi_t|^2\right)\frac{\varphi}{\vartheta}\,\mathrm{d}x\,\mathrm{d}r - \int_s^t\int_{\partial\Omega} h\frac{\varphi}{\vartheta}\,\mathrm{d}S\,\mathrm{d}r,
\end{aligned}$$

where φ is a sufficiently regular, *positive* test function, coupled with the following *total energy inequality*

$$\mathcal{E}(\vartheta(t),\mathbf{u}(t),\mathbf{u}_t(t),\chi(t)) \leq \mathcal{E}(\vartheta(s),\mathbf{u}(s),\mathbf{u}_t(s),\chi(s)) + \int_s^t\int_\Omega g\,\mathrm{d}x\,\mathrm{d}r + \int_0^t\int_{\partial\Omega} h\,\mathrm{d}S\,\mathrm{d}r + \int_s^t\int_\Omega \mathbf{f}\cdot\mathbf{u}_t\,\mathrm{d}x\,\mathrm{d}r\,, \qquad (11)$$

where

$$\mathcal{E}(\vartheta,\mathbf{u},\mathbf{u}_t,\chi) := \int_\Omega \vartheta\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\mathbf{u}_t|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega b(\chi(t))\varepsilon(\mathbf{u}(t))\mathbb{E}\varepsilon(\mathbf{u}(t))\,\mathrm{d}x + \frac{1}{p}\int_\Omega |\nabla\chi|^p\,\mathrm{d}x + \int_\Omega W(\chi)\,\mathrm{d}x\,. \qquad (12)$$

Both (10) and (11) are, in the general case, required to hold for almost all $t \in (0,T]$ and almost all $s \in (0,t)$, and for $s = 0$. This notion of solution corresponds exactly to the physically meaningful requirement that the system should satisfy the second and first principle of Thermodynamics. Indeed, one of the main advantages of this formulation resides in the fact that the thermodynamically consistency of the model immediately follows from the existence proof. It can be also shown that this solution concept is consistent with the standard one.

From an analytical viewpoint, observe that the entropy inequality (10) has the advantage that all the troublesome quadratic terms on the right-hand side of (1) feature as multiplied by a negative test function. This, and the fact that (10) is an inequality, allows for upper semicontinuity arguments in the limit passage in a suitable approximation of (10)–(12).

In addition to (10)–(12), the *"entropic formulation"* of system (1)–(3) also consists of the momentum balance (2), given pointwise a.e. in $\Omega \times (0, T)$, and of the internal variable equation (3). The latter is required to hold pointwise almost everywhere in the reversible case $\mu = 0$. In the irreversible case $\mu = 1$, we shall confine the analysis to the case in which $\widehat{\beta}$ is the indicator function $I_{[0,+\infty)}$ of $[0, +\infty)$, hence $W(\chi) = I_{[0,+\infty)}(\chi) + \widehat{\gamma}(\chi)$. We shall have to weakly formulate (3) in terms of the requirement $\chi_t \leq 0$ a.e. in $\Omega \times (0, T)$, of the *one-sided variational inequality*

$$\int_\Omega \left(\chi_t - \operatorname{div}(|\nabla\chi|^{p-2}\nabla\chi) + \xi + \gamma(\chi) + b'(\chi)\frac{\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u})}{2} - \vartheta \right) \psi \, \mathrm{d}x \geq 0 \text{ for all } \psi \in W^{1,p}(\Omega) \text{ with } \psi \leq 0, \tag{13}$$

almost everywhere in $(0, T)$ (where $\gamma := \widehat{\gamma}'$), and of the *energy-dissipation inequality (for the internal variable χ)*

$$\begin{aligned} \int_s^t \int_\Omega |\chi_t|^2 \, \mathrm{d}x \, \mathrm{d}r + \int_\Omega \left(\frac{1}{p} |\nabla\chi(t)|^p + W(\chi(t)) \right) \mathrm{d}x \\ \leq \int_\Omega \left(\frac{1}{p} |\nabla\chi(s)|^p + W(\chi(s)) \right) \mathrm{d}x + \int_s^t \int_\Omega \chi_t \left(-b'(\chi)\frac{\varepsilon(\mathbf{u})\mathbb{E}\varepsilon(\mathbf{u})}{2} + \vartheta \right) \mathrm{d}x \, \mathrm{d}r \end{aligned} \tag{14}$$

for all $t \in (0, T]$ and almost all $s \in (0, t)$, with ξ a selection in the (convex analysis) subdifferential $\partial\widehat{\beta}(\chi) = \partial I_{[0,+\infty)}(\chi)$ of $I_{[0,+\infty)}$. In [12, Prop. 2.14], we prove that, under additional regularity properties any weak solution in fact fulfills (3) pointwise.

The existence results. The main results of [13] state the existence of "*entropic solutions*" for system (1–3), supplemented with the boundary conditions (4), in the irreversible ($\mu = 1$) and reversible ($\mu = 0$) cases.

More precisely, in the case of unidirectional evolution for χ we can prove the existence of a global-in-time "*entropic solution*" (i.e. satisfying the *entropy* (10) and the *total energy* (11) inequalities, the (pointwise) momentum balance (2), the *one-sided variational inequality* (13) and the *energy* (14) inequalities for χ). We work under fairly general assumptions on the nonlinear functions in (1)–(3). More precisely, we require that a and b are sufficiently smooth and bounded from below by a positive constant, b convex, and we standardly assume that $W = I_{[0,+\infty)} + \widehat{\gamma}$, with $\widehat{\gamma}$ smooth and λ-convex. A crucial role is played by the requirement that the heat conductivity function $\mathsf{K} = \mathsf{K}(\vartheta)$ grows at least like ϑ^κ with $\kappa > 1$.

Moreover, under some restriction on κ (i.e. $\kappa \in (1, 5/3)$ for space dimension $d = 3$), we can also obtain an enhanced regularity for ϑ and conclude that the *total energy inequality* actually holds as an *equality*.

In the reversible case ($\mu = 0$), instead, under the same assumptions described above (but with a general $\widehat{\beta}$), we improve the estimates, hence the regularity, of the internal variable χ. Therefore, we prove the existence of a weak formulation of (1)–(3), featuring, in addition to (10), (11), and (2), a *pointwise* formulation of equation (3). Again, in the case of the aforementioned restriction on κ, we enhance the time-regularity of ϑ. Moreover, exploiting the improved formulation of the equation for χ, we conclude existence for a stronger formulation of the heat equation (2), of variational type. Unfortunately, a uniqueness result seems to be out of reach, at the moment, not only in the irreversible but also in the reversible cases. Only for the *isothermal* reversible system a continuous dependence result, yielding uniqueness, can be proved exactly like in [12, Thm.3].

Finally, we address the analysis of system (1)–(3), with $\mu = 1$, in the case the p-Laplacian regularization in (3) is replaced by the standard Laplacian operator. We approximate it by adding a p-Laplacian term, modulated by a small parameter δ, on the left-hand side of (3), so that we get the existence of approximate solutions $(\vartheta_\delta, \mathbf{u}_\delta, \chi_\delta)$. Then, we let δ tend to zero. In this context, the enhanced elliptic regularity estimates on the momentum equation which would here yield some suitable compactness for the quadratic term $a(\chi_\delta)\varepsilon(\partial_t \mathbf{u}_\delta)\mathbb{V}\varepsilon(\partial_t \mathbf{u}_\delta)$ on the right-hand side of (1), are no longer available. In fact, they rely on the requirement $p > d$. A crucial step for proving the existence of (a slightly weaker notion of) "*entropic solutions*" to system (1)–(3), then consists in deriving some suitable strong convergence for $(\partial_t \mathbf{u}_\delta)_\delta$ with an ad hoc technique, strongly relying on the fact that $\mu = 1$, and on the additional assumption that b is non-decreasing.

Our main existence results are proved by passing to the limit in a time-discretization scheme, unique for the reversible and the irreversible cases, carefully tuned to the nonlinear features of the PDE system. In particular, it is devised in such a way as to obtain that the piecewise constant and piecewise linear interpolants of the discrete solutions satisfy the discrete versions of the entropy inequality (10), of total energy inequality (12), and of the energy inequality (14) in the case $\mu = 1$. Moreover, with delicate calculations we are also able to translate to the time-discrete level a series of a priori estimates on the heat equation, having a nonlinear character. Note that this detailed time-discrete analysis could be interesting in view of further numerical studies of this model.

3.2 The main results on two-phase fluids flows

Main assumptions. For the sake of simplicity, we take $\varepsilon = 1$ in (6) and (8). Next, we assume that $F(\varphi)$ is the classical double-well potential, namely $F(\varphi) = \frac{1}{4}(\varphi^2 - 1)^2$, so that $F'(\varphi) = \varphi^3 - \varphi$. More general expressions of F' having cubic growth at ∞ may be admissible as well, but we prefer to treat only the classical double well case in order not to overburden the presentation. We assume that the thermal conductivity, the specific heat and the viscosity of the mixture depend on θ in the following way:

$$\kappa(\theta) = 1 + \theta^{\beta}, \qquad c_V(\theta) = \theta^{\delta}, \qquad 0 < \underline{\nu} \le \nu(\theta) \le \overline{\nu}, \tag{15}$$

for all $\theta \ge 0$, some $0 < \underline{\nu} < \overline{\nu}$, and some $\beta > 0, \delta > 0$ complying with the following restrictions: $\beta \ge 2$, $\frac{1}{2} < \delta < 1$. In view of (15), we can compute

$$Q(\theta) = \int_0^{\theta} c_V(s)\,\mathrm{d}s = \frac{1}{\delta+1}\theta^{\delta+1}, \quad \Lambda(\theta) = \frac{1}{\delta}(\theta^{\delta} - 1). \tag{16}$$

The ansatz $\delta < 1$ actually implies that the thermal component Λ of the entropy is concave, as prescribed by Thermodynamics, while the other limitations mainly have a mathematical motivation and are needed in order to obtain the necessary regularity to pass to the limit. Notice however that a power-like behavior for the heat-conductivity is typical of several types of fluids. Regarding instead the initial data, we assume that:

$$\mathbf{v}_0 \in L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3), \qquad \varphi_0 \in H^1(\Omega), \qquad \theta_0 \in L^{\delta+1}(\Omega), \quad \theta_0 > 0 \text{ almost everywhere.} \tag{17}$$

Here and below, L^2_{div} indicates the space of divergence-free L^2 functions.

The "entropic formulation". First of all, we rewrite the momentum equation (6) in the more explicit form:

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla_x\mathbf{v} + \nabla_x p = \operatorname{div}\mathbb{S} - \operatorname{div}(\nabla\varphi\otimes\nabla\varphi). \tag{18}$$

This permits us to introduce the notion of "*entropic solution*" to our model problem: An "*entropic solution*" to the non-isothermal diffuse interface model for two-phase flows of fluids is a quadruplet $(\mathbf{v},\varphi,\mu,\theta)$ satisfying the *incompressibility* condition $\operatorname{div}\mathbf{v} = 0$ a.e. in $(0,T)\times\Omega$, the *weak momentum balance*

$$\int_0^T\int_\Omega(\mathbf{v}\cdot\partial_t\boldsymbol{\xi} + (\mathbf{v}\otimes\mathbf{v}):\nabla_x\boldsymbol{\phi} + p\operatorname{div}\boldsymbol{\xi}) = \int_0^T\int_\Omega(\mathbb{S}:\nabla_x\boldsymbol{\xi}) - \int_0^T\int_\Omega(\nabla\varphi\otimes\nabla\varphi):\nabla_x\boldsymbol{\xi} - \int_\Omega\mathbf{v}_0\cdot\boldsymbol{\xi}(0,\cdot), \tag{19}$$

for all $\boldsymbol{\xi} \in \mathcal{C}^\infty_0([0,T)\times\overline{\Omega};\mathbb{R}^3)$ such that $\boldsymbol{\xi}\cdot n_{|\Gamma} = 0$, the *Cahn-Hilliard system*

$$\langle\varphi_t,\xi\rangle + \int_\Omega(\mathbf{v}\cdot\nabla_x\varphi)\xi = \int_\Omega\nabla\mu\cdot\nabla\xi \qquad \text{for all } \xi\in V, \text{ and a.e. in } (0,T), \tag{20}$$

$$\mu = -\Delta\varphi + F'(\varphi) - \theta \qquad \text{a.e. in } (0,T)\times\Omega, \tag{21}$$

with the boundary condition

$$\nabla_x\varphi\cdot n_{|\Gamma} = 0 \tag{22}$$

and the initial condition $\varphi(0,\cdot)=\varphi_0$, the *weak total energy balance*

$$\int_0^T\int_\Omega\left(\frac{1}{2}|\mathbf{v}|^2+e\right)\partial_t\xi+\int_0^T\int_\Omega\left(\frac{1}{2}|\mathbf{v}|^2\,\mathbf{v}+e\,\mathbf{v}\right)\cdot\nabla_x\xi+\int_0^T\int_\Omega\hat{\kappa}(\theta)\Delta\xi+\int_0^T\int_\Omega p\mathbf{v}\cdot\nabla\xi$$
$$-\int_0^T\int_\Omega(\mathbb{S}\,\mathbf{v})\cdot\nabla_x\xi+\int_0^T\int_\Omega\frac{\mu^2}{2}\Delta\xi+\int_0^T\int_\Omega(\mathbf{v}\cdot\nabla_x\varphi)\,(\nabla_x\varphi\cdot\nabla_x\xi)+\int_0^T\int_\Omega(\nabla_x\mu\otimes\nabla_x\xi):\nabla_x\nabla_x\varphi$$
$$+\int_0^T\int_\Omega(\nabla_x\mu\otimes\nabla_x\varphi):\nabla_x\nabla_x\xi-\int_\Omega\left(\frac{1}{2}|\mathbf{v}_0|^2+e_0\right)\xi(0,\cdot)=0,\quad\text{for all }\xi\in\mathcal{C}_0^\infty([0,T)\times\Omega),\tag{23}$$

and the following weak form of the *entropy production inequality*:

$$\int_0^T\int_\Omega(\Lambda(\theta)+\varphi)\,\partial_t\xi+\int_0^T\int_\Omega(\Lambda(\theta)+\varphi)\,\mathbf{v}\cdot\nabla_x\xi+\int_0^T\int_\Omega h(\theta)\Delta\xi$$
$$\le-\int_0^T\int_\Omega\left(\frac{\nu(\theta)}{\theta}|\nabla_x\mathbf{v}|^2+\frac{1}{\theta}|\nabla_x\mu|^2+\frac{\kappa(\theta)}{\theta^2}|\nabla_x\theta|^2\right)\xi-\int_\Omega(\Lambda(\theta_0)+\varphi_0)\cdot\xi(0,\cdot),\tag{24}$$

holding for any $\xi\in\mathcal{C}_0^\infty([0,T)\times\Omega)$, $\xi\ge 0$ and where we have set $h(\theta)=\int_1^\theta\frac{\kappa(s)}{s}\,\mathrm{d}s=\log\theta+\frac{1}{\beta}(\theta^\beta-1)$, e is given by $e=\frac{1}{\varepsilon}F(\varphi)+\frac{\varepsilon}{2}|\nabla\varphi|^2+Q(\theta)$, $\hat{\kappa}$ is defined as $\hat{\kappa}(\theta)=\int_0^\theta\kappa(s)\,\mathrm{d}s=\theta+\frac{1}{\beta+1}\theta^{\beta+1}$, and $e_0=F(\varphi_0)+\frac{|\nabla_x\varphi_0|^2}{2}+Q(\theta_0)$.
It is worth noting that (19) incorporates both the incompressibility constraint and the initial condition $\mathbf{v}(0,\cdot)=\mathbf{v}_0$; moreover it accounts for the complete-slip conditions

$$\mathbf{v}\cdot n_{|\Gamma}=0,\qquad[\mathbb{S}n]\times n_{|\Gamma}=0.\tag{25}$$

The first equation (20) of the Cahn-Hilliard system is in weak form accounting for the no-flux condition

$$\nabla_x\mu\cdot n_{|\Gamma}=0,\tag{26}$$

while we will be able to prove sufficient regularity on φ in order for (21) to hold pointwise (with the no-flux condition (22) in the sense of traces).

Existence of global-in-time solutions. The main result of [2] reads as follows: under the assumptions stated above, the non-isothermal diffuse interface model for two-phase flows of fluids admits at least an "*entropic solution*" in the following regularity class:

$$\mathbf{v}\in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))\cap L^2(0,T;\mathbf{V_n})\tag{27}$$
$$\varphi\in H^1(0,T;(H^1(\Omega))')\cap L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^3(\Omega))\tag{28}$$
$$\mu\in L^2(0,T;H^1(\Omega))\cap L^{\frac{14}{5}}((0,T)\times\Omega)\tag{29}$$
$$\theta\in L^\infty(0,T;L^{\delta+1}(\Omega))\cap L^\beta(0,T;L^{3\beta}(\Omega))\cap L^2(0,T;H^1(\Omega))\tag{30}$$
$$\theta>0\ \text{ a.e. in }(0,T)\times\Omega,\quad\log\theta\in L^2(0,T;H^1(\Omega)),\tag{31}$$

δ and β being specified above. Let us notice that in case we could prove existence of a *sufficiently smooth* weak solution (in particular, regular enough in order to integrate back by parts the terms in (23)), then it would be possible to show that such a solution also satisfies the "standard" form of the heat equation (9). Equivalently, the entropy inequality would hold as an equality in that case. Hence, the current notion of weak solution turns out to be compatible both with Thermodynamics and also with the "strong" one.

4 Conclusions

Other approaches to the weak solvability of coupled PDE systems with an L^1-right-hand side are available in the literature: in particular, we can refer to [15] and [14]. In [15], the notion of *renormalized solution* has been used in order to prove a global-in-time existence result for a nonlinear system in thermoviscoelasticity, while [14] focuses on rate-independent processes coupled with viscosity and inertia in the displacement equation, and with the temperature equation. There the internal variable equation features a 1-positively homogeneous dissipation potential. For the resulting PDE system, a weak solution concept partially mutuated from the theory of rate-independent processes by A. Mielke (cf., e.g., [11]) is studied. An existence result is proved combining techniques for rate-independent evolution, with Boccardo-Gallouët type estimates of the temperature gradient in the heat equation with L^1-right-hand side.

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